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ORIGINAL RESEARCH

Kurt Bruckner's view on the Penrose tiling

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Abstract We demonstrate the potential of Kurt Bruckner's 'addition algorithm', which is based on the substitution rule for the generation of the Robinson triangle tiling, a variant of the Penrose tiling. The artist Kurt Bruckner developed his straightforward approach intuitively for the creation of quasiperiodic ornaments. This versatile method can be used for the construction of achiral, homochiral and racemic quasiperiodic ornaments, as well as for the generation of decorated two-level (two-color) Penrose tilings. Therefore, the underlying tiling is always the same kind of Penrose tiling, which is invariant under the action of specific mirror and black/white mirror operations in contrast to unit tiles that are decorated in specific ways. Compared to the underlying classical substitution method the advantage of Kurt Bruckner's approach is its simplicity and versatility for the creation of decorated tilings. Using a vector graphics editor, large and arbitrarily complex quasiperiodic ornaments can be easily generated manually.

Keywords Quasiperiodic · Fivefold symmetry · Penrose tiling · Robinson triangles · Chiral · Ornaments

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Introduction

This topical issue is dedicated to Alan L. Mackay on the occasion of his 90th birthday. How is our contribution linked to his work? Alan L. Mackay was showing interest in fivefold symmetry already long before Dan Shechtman's seminal discovery of icosahedral quasicrystals in April 1982. His paper on the icosahedral packing of equal spheres appeared already 20 years earlier [1]. Shortly after the first publication of a two-dimensional quasiperiodic tiling with local fivefold symmetry by Roger Penrose [2], later called Penrose tiling (PT), Alan L. Mackay discussed at the 10th Conference of the Yugoslav Centre of Crystallography recursive rules for building pentagons and triangles based on two isosceles triangles (Robinson triangles) [3]. A few years after the popularization of the PT by Martin Gardner's article in the journal 'Scientific American' [4], Alan L. Mackay was the first to show by optical diffraction how a diffraction pattern of a PT should look like [5]. Had Dan Shechtman known about his work, his discovery of quasicrystalline Al–Mn would not have taken more than 2 years to get published [6]. For an introduction into the topics of tilings and quasicrystals in general see, for instance, the textbook 'Crystallography of quasicrystals' by Steurer and Deloudi [7] and the comprehensive monograph 'Tilings and Patterns' by Grünbaum and Shephard [8].

The sculptor, painter and graphic artist Kurt Bruckner (KB) was inspired by the Penrose tiling to create beautiful ornamental patterns; some of them are shown in Fig. 1. Moreover, he developed an 'addition algorithm', a remarkably simple way to generate his decorated tilings.

Born in 1953 in Styria, Austria, KB attended the Art Academy Brera in Milan after completing his training as a sculptor. Since 1982, he has been working as sculptor in

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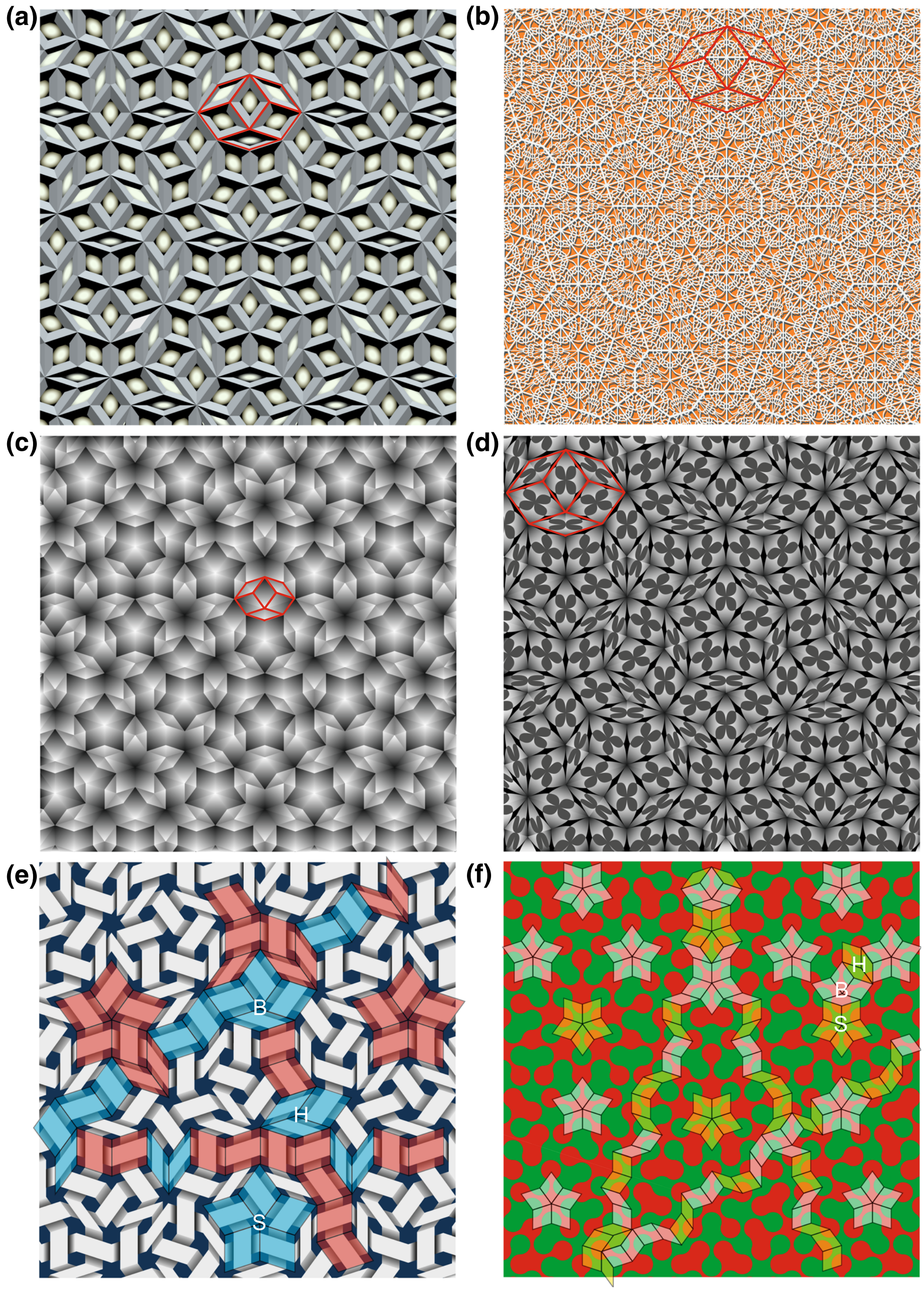


Fig. 1 Examples of Kurt Bruckner's ornamental creations (**a** 100617 1, **b** 150918, **c** 150925, **d** 141124, **e** 121213, **f** 111228. The six digits give the year/month/day in which they were created.) with the unit tiles and some structure motifs marked by the authors. Subfigures **a–d** are achiral decorations of the regular PT, while the ones in (**e**) and (**f**) are two-level PTs. The one in subfigure (**e**) corresponds to a racemic tiling of chiral unit tiles, and the one in (**f**) to a two-level PT. Along a 'worm' all fat rhombs have the same handedness (*marked blue or red*) or *color* distribution (*marked yellow or white*) until it is changed by a single skinny rhomb. Two subsequent skinny rhombs leave it unchanged. (Color figure online)

Schaffhausen, Switzerland. His specialty is concrete casting. In 2001, in the course of an artistic reorientation he became interested in ornamentation, initially mainly based on Arabic ornaments. In 2004, he came by chance across a picture of the PT in the popular journal 'Scientific American.' KB was thrilled with the quasiperiodic patterns in which he saw a great potential for the development of his own artistic ornaments. Without deeper mathematical knowledge, just by visual inspection, he developed his own approach for the manual generation of the PT with arbitrary decorations. Examining the resulting complex interlace patterns, he was fascinated by the fact that rather than the translational symmetry of the familiar traditional ornaments here the fivefold rotational symmetry conveys the dynamism of the narrative of the image. KB has been experimenting with many different ways of decorating the underlying quasiperiodic patterns, because he is primarily interested in the ornamental as an artistic principle of visual form and image generation. With his ornamental patterns he wants to challenge viewers in their perception by irritation. His esthetic approach to art—and on side roads to mathematics—is of playful nature, searching, discovering, combining, experimenting. For more information on his work see, for instance, his website http://kurtbruckner.ch/.

Kurt Bruckner's 'addition algorithm'

KB's 'addition algorithm' can be related to the classic substitution rule for the generation of the PT in the version of the Robinson triangle tiling. In Fig. 2, it is shown how the two prototiles (Fig. 2a), the isosceles triangles A^0 and B^0, have to be combined in their original and reflected form, respectively. In the first step, the two unit Penrose rhomb tiles are formed (Fig. 2b). After each substitution operation, the resulting patches of tiles with the shape of the original unit tiles are inflated by a factor τ (Fig. 3e). The number $\tau = 2\cos\pi/5 = \left(1+\sqrt{5}\right)/2 = 1.618\cdots$ is an algebraic irrational number, i.e., the solution of the equation $\tau^2 - \tau - 1 = 0$, called the golden mean or golden ratio. The Ammann line segments on the unit tiles are going to form a Fibonacci pentagrid on the growing tiling proving its quasiperiodicity in this way.

In the following, we will present KB's 'addition algorithm' step by step (Fig. 3):

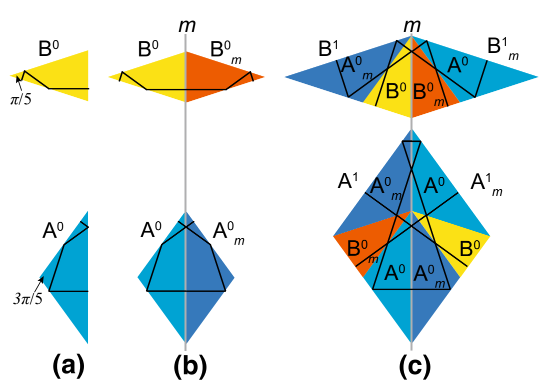


Fig. 2 Generation of the Robinson triangle tiling based on the substitution rule. The isosceles triangles (**a**) are marked by Ammann lines in order to show the matching rules. In the PT, the decorated Robinson triangles occur each in both enantiomorphs. *Yellow/orange* and *light blue/dark blue triangle* pairs are each related by mirror symmetry. The *different colors* of the reflected unit tiles just indicate the result of a reflection operation m and should not be misinterpreted as resulting from a *color* reflection m'. The skinny and fat Penrose unit rhomb tiles (**b**) consist each of the respective two mirror symmetric Robinson triangles. The first inflation of the unit rhombs is shown in (**c**). One sees that the inflated fat rhomb tile also contains the inflated skinny rhomb tile. The term 'inflation' refers to the increased number of unit tiles as well as to the blown-up dimension of the resulting rhomb tiles. (The *drawing* is based on the respective figure in the *Tilings Encyclopedia*, http://tilings.math.uni-bielefeld.de/substitution_rules/robinson_triangle.) (Color figure online)

- We start with the two kinds of isosceles Robinson triangles A (blue) and B (yellow), both with acute angles $\pi/5$ and an area ratio A/B of τ, which may be arbitrarily decorated (Fig. 3a). In our case, we decorate them with Ammann line segments in order to confirm the quasiperiodicity of the resulting tiling. Only if the segments form five sets of straight lines with particular properties, the tiling is strictly quasiperiodic.
- In the first step, we combine the Robinson triangles to the skinny (*alias* prolate, thin; yellow) and the fat (*alias* oblate, thick; blue) Penrose rhomb unit tiles, respectively (zeroth generation, $n = 0$). This can be done by either a reflection operation m (Figs. 3b, 5b), or a color (black/white) reflection operation m' (Fig. 4b), i.e., an operation that not only reflects a motif but also changes its color in a defined way (from black to white, red to green, blue to yellow, etc.).
- Then, we attach on top of the fat rhomb the two reflected and rotated halves of the skinny and the fat rhombs of the zeroth generation (Fig. 3c–e). The half original fat rhomb A plus the attached half skinny rhomb B together give the half skinny rhomb (Robinson triangle B^1) of the first generation ($n = 1$). With the further addition of a half original fat rhomb A, one gets the half fat rhomb (Robinson triangle A^1) of the first generation ($n = 1$).
- Subsequently, we attach to the fat rhomb of the first generation its two reflected and rotated halves (Robinson triangles A_m^1), but now at the bottom. The gaps are filled by the halves of skinny rhombs (Robinson triangle B_m^1) of the first generation, which are extracted from reflected subregions of the halves of the fat rhombs (Robinson triangles A_m^1) (Fig. 3h).
- We repeat the previous steps again and again each time with patches on a τ larger scale. Note that the attachment of the reflected fat rhomb halves takes place at the top and at the bottom of the fat rhombs alternatingly.

KB's algorithm always generates the same PT underlying the various ornamental patterns; the only difference in symmetry results from the symmetry of the decoration of the unit tiles. If the decorated rhomb unit tiles have the holohedral symmetry $2mm$, the resulting tiling has the same symmetry as the non-decorated tiling itself. In the case of a color-chiral decoration of the rhombs, we can get a homochiral PT (Fig. 4j–r) or a racemic PT (Figs. 1e, 4a–i). Replacing the regular mirror lines m by color-(black/white)-mirror lines m', we obtain a two-level (two-color) PT (Figs. 1f, 5). In Fig. 1e, f, a few so-called 'worms' are marked. Such a 'worm' consists of a sequence of rhombs, each with two edges perpendicular to its propagation

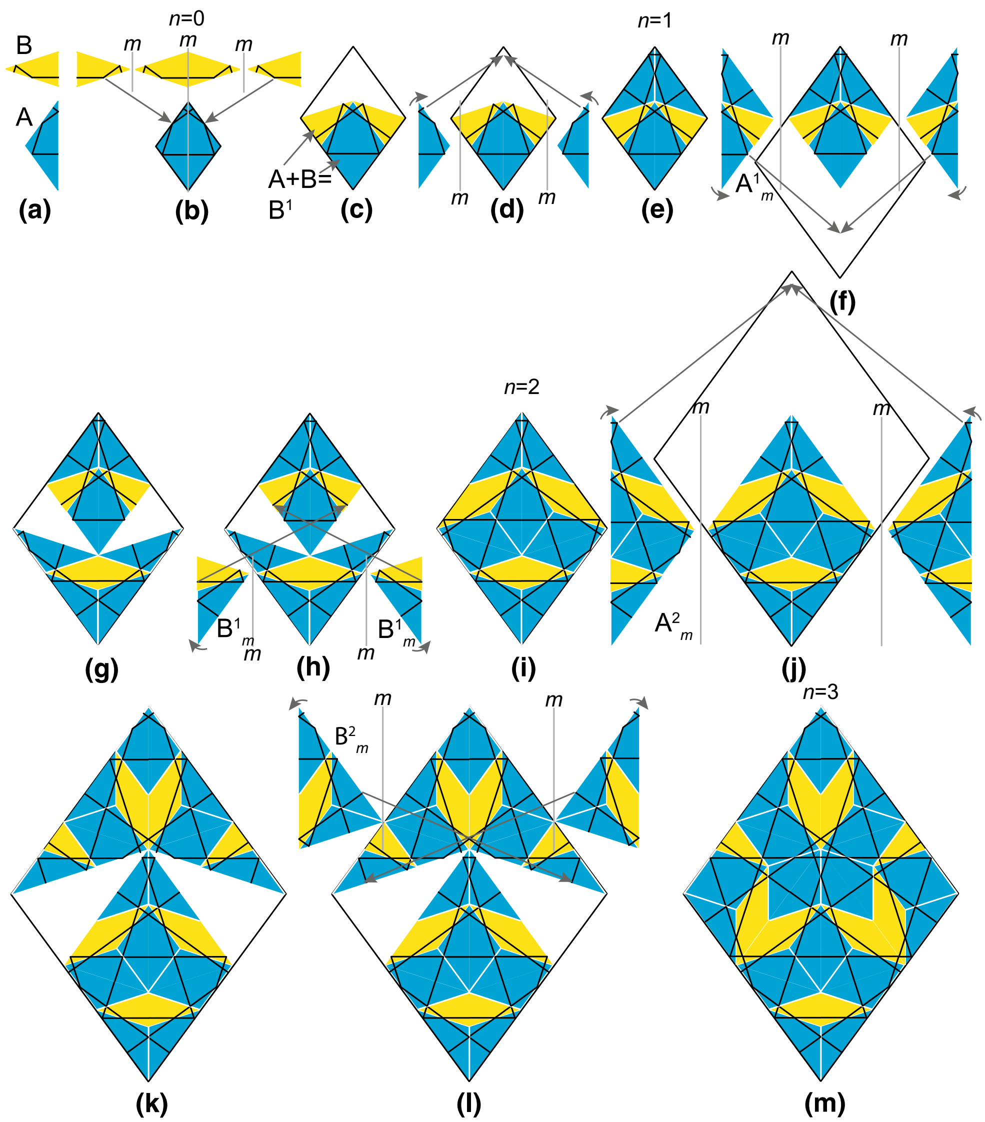


Fig. 3 Kurt Bruckners 'addition algorithm' for the creation of the rhomb Penrose tiling (see main text). Instead of substitutions, he applies reflections and rotations. For proving the quasiperiodicity of the resulting tiling, the unit tiles are decorated with Ammann line segments. These connect to sets of *straight lines* when a PT is formed

direction. Each single rhomb of the PT is in the intersection of two each other crossing 'worms.' Several 'boat' (B), 'hexagon' (H) and 'star' (S) supertiles are marked, which consist of unit tiles of the same handedness/color distribution.

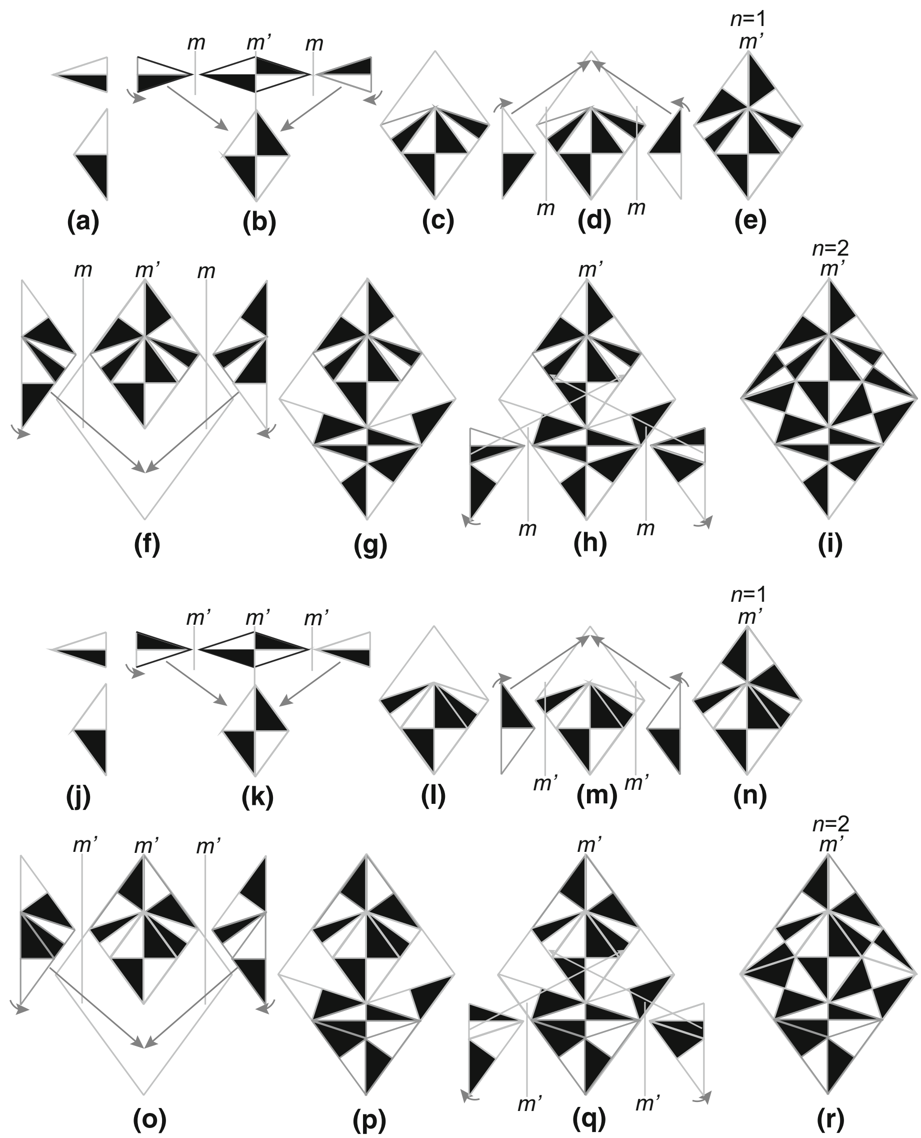
Each inflation step leads to a PT with the shape of the fat Penrose unit rhomb and with the total area increased by a factor of τ^2. The number of fat and skinny rhombs for the nth inflation step corresponds to the Fibonacci numbers F_{2n+1} and F_{2n}, respectively. The Fibonacci numbers $F_n = 0, 1, 1, 2, 3, 5, 8, 13, \ldots$. are defined by the recursive equation $F_{n+1} = F_n + F_{n-1}$, with $F_0 = 0$ and $F_1 = 1$.

Symmetry of the decorated tilings

It has to be kept in mind that the KB algorithm, applied to empty Robinson triangles, gives always a patch of the PT with the shape of the fat Penrose rhomb and the point group symmetry *m*. Depending on the symmetry of the decoration of the unit rhombs and the kind of reflection operation applied (m or m'), the symmetry of the resulting patch of the PT can be m or m'. The application of fivefold rotation to the final inflated pattern of the PT leads to a five-star with symmetries $5m$ or $5m'$, respectively.

If the starting unit rhombs are decorated in a mirror symmetric way around the long (short) body diagonal of

Fig. 4 If we apply *black/white* reflection operations m' for creating chiral Penrose unit tiles from Robinson triangles, which already have b/w mirror symmetry, and just simple reflections m later on, the finally resulting inflated fat Penrose unit rhomb will show symmetry m' as well (**a**)–(**i**). We get a racemic tiling where both polymorphs will be equally distributed finally. If we apply m' operations only in the inflation process, then the finally resulting inflated fat Penrose unit rhomb will show symmetry m' and will be homochiral (**j**)–(**r**). All unit tiles will have the same handedness as the original ones



the fat (skinny) rhomb, then the resulting symmetry after each inflation step, which is obtained by applying just mirror operations m, remains m.

If the starting unit rhombs are decorated in a chiral way, e.g., with black/white (b/w) mirror symmetry m' around the long and short body diagonals of the unit rhombs, then the resulting symmetry after each inflation step, which is obtained by applying just mirror operations m, remains m'. However, the inflated patterns are racemic, i.e., they contain both left- and right-handed copies of the original unit rhombs (Figs. 1e, 4a–i). Their frequencies for the nth inflation step corresponds to the Fibonacci numbers F_{2n} and F_{2n-1}, respectively, for the fat rhombs and F_{2n-1} and F_{2n-2}, respectively, for the skinny rhombs. The number of copies of the original handedness is F_{2n} in case of the fat rhombs and F_{2n-2} in case of the skinny rhombs.

If we use b/w mirror lines m' for the inflation process starting from b/w chiral fat and skinny rhombs, we get inflated patterns, which will be homochiral due to $m' \cdot m' = 1$; the symmetry of the patch will be m' and frequencies of the fat and skinny rhombs F_{2n+1} and F_{2n}, respectively (Fig. 4j–r).

In case of mirror symmetric unit tiles and b/w mirrors m' operating in the inflation process, the two-color (two-level) PT [9] with symmetry m results (Figs. 1f, 5). Patches of unit tiles of the same color form the H, B, S supertiles. Their frequencies are in a ratio $n_H : n_B : n_S = \tau\sqrt{5} : \sqrt{5} : 1$. Tilings based on these three supertiles are called HBS tilings. They are dual to the Penrose pentagon tiling.

Caveat: The above symmetry considerations refer only to the symmetry of the finite patches of the PT generated by

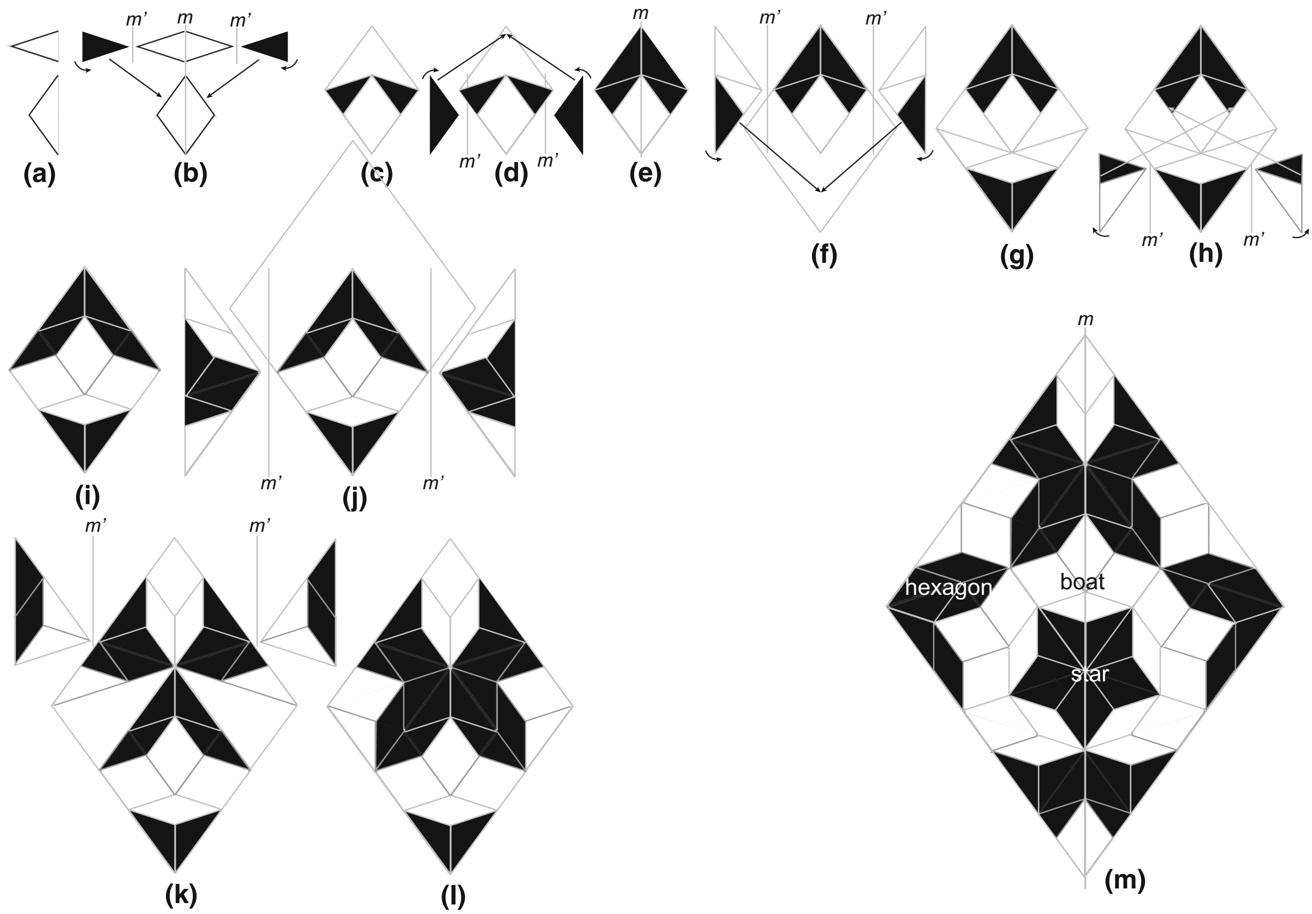


Fig. 5 If we apply a *black/white* reflection operation m' for creating the Penrose unit tiles from Robinson triangles with mirror symmetry and also *black/white* reflection operations m' later on, the finally resulting inflated fat Penrose unit rhomb will show symmetry m as well. The tiling itself will be a *black/white* or two-level Penrose tiling (see [8], and references therein)

the KB algorithm. Although there is no point of local tenfold symmetry in a regular PT (only in the exceptionally singular PT; see [10]), the point group symmetry of an infinite PT is decagonal, 10*mm*, and that of the two-color PT, for instance, is 10'$m'm$ (see, e.g., [11]). The decagonal symmetry is reflected in the 'bond-orientational order,' i.e., the orientational distribution of the tile edges in the PT. It is also reflected in the symmetry of its Fourier image, i.e., the distribution of structure factors in the Fourier space (see, e.g., [12], Figs. 4.6.3.24–27).

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